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Weak convergence theorems for common solutions of a system of equilibrium problems and operator equations involving nonexpansive mappings

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Abstract

In this paper, a monotone variational inequality, a system of equilibrium problems, and nonexpansive mappings are investigated based on an iterative algorithm. Weak convergence theorems of common solutions are established in Hilbert spaces.

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Keywords: equilibrium problem; monotone mapping; nonexpansive mapping; projection; variational inequality

1 Introduction

Equilibrium problems which were introduced by Blum and Oettli [1] have intensively been studied. It has been shown that equilibrium problems cover fixed point problems, variational inequality problems, inclusion problems, saddle problems, complementarity problem, minimization problem, and Nash equilibrium problem; see [1–3] and the references therein. Equilibrium problem has emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization; see [4–7] and the references therein. For the existence of solutions of equilibrium problems, we refer the readers to [8–13] and the references therein. However, from the standpoint of real world applications, it is important not only to know the existence of solutions of equilibrium problems, but also to be able to construct an iterative algorithm to approximate their solutions. The computation of solutions is important in the study of many real world problems. For instance, in computer tomography with limited data, each piece of information implies the existence of a convex set in which the required solution lies. The problem of finding a point in the intersection of a finite of the convex sets is then of crucial interest and it cannot be directly solved. Therefore, an iterative algorithm must be used to approximate such a point. The well-known convex feasibility problem which captures applications in various disciplines such as image restoration and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonlinear mappings; see, for example, [14–19].

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In this paper, a monotone variational inequality, a system of equilibrium problems, and nonexpansive mappings are investigated based on an iterative algorithm. Weak convergence theorems of common solutions are established in Hilbert spaces.

2 Preliminaries

In what follows, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and C is a nonempty closed and convex subset of H.

Let $\mathbb{R}$ denote the set of real numbers and F a bifunction of $C \times C$ into $\mathbb{R}$. Recall the bifunction equilibrium problem is to find an x such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

In this paper, the solution set of the equilibrium problem is denoted by $EP(F)$, *i.e.*,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}.$$

To study the equilibrium problems (2.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;
(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to stand for the set of fixed points. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x,y \in C.$$

If C is a bounded closed and convex subset of H, then fixed point sets of nonexpansive mappings are not empty, closed, and convex; see [20] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x,y \in C.$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y,g) \in G(T)$ implies $f \in Tx$. The class of monotone operators is one of the most important classes of operators. Within the past several decades, many authors have been devoted to the studies on the existence and convergence of zero points for maximal monotone operators.

Let $F(x,y) = \langle Ax, y-x\rangle$, $\forall x, y \in C$. We see that the problem (2.1) is reduced to the following classical variational inequality. Find $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

It is known that $x \in C$ is a solution to (2.2) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant and I is the identity mapping.

Recently, the common solution problems have been extensively studied by many scholars; see, for example, [21–33] and the references therein. In this paper, we investigate the common solution problem of a monotone variational inequality, a system of equilibrium problems, and nonexpansive mappings based on an iterative algorithm. In order to prove our main results, we need the following lemmas.

Lemma 2.1 *Let C be a nonempty closed and convex subset of H. Then the following inequality holds*:

$$\|x - \mathrm{Proj}_C x\|^2 + \|y - \mathrm{Proj}_C y\|^2 \leq \|x-y\|^2, \quad \forall x \in H, y \in C.$$

Lemma 2.2 [1, 2] *Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued*;

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = EP(F)$*;

(d) *$EP(F)$ is closed and convex.*

Lemma 2.3 [33] *Let A be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \{ w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C \}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \geq 0$ for all $u \in C$.

Lemma 2.4 [34] *Let* $\{a_n\}_{n=1}^N$ *be real numbers in* $[0,1]$ *such that* $\sum_{n=1}^N a_n = 1$. *Then we have the following*:

$$\left\| \sum_{i=1}^N a_i x_i \right\|^2 \le \sum_{i=1}^N a_i \|x_i\|^2,$$

for any given bounded sequence $\{x_n\}_{n=1}^N$ *in* H.

Lemma 2.5 [35] *Let* $0 < p \le t_n \le q < 1$ *for all* $n \ge 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* H *such that*

$$\limsup_{n\to\infty} \|x_n\| \le d, \qquad \limsup_{n\to\infty} \|y_n\| \le d$$

and

$$\lim_{n\to\infty} \left\| t_n x_n + (1-t_n) y_n \right\| = d$$

hold for some $r \ge 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6 [36] *Let* C *be a nonempty closed and convex subset of* H *and* $S : C \to C$ *be a nonexpansive mapping. If* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x$ *and* $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, *then* $x = Sx$.

Lemma 2.7 [37] *Let* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \le (1 + b_n) a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 *is some nonnegative integer,* $\sum_{n=1}^\infty b_n < \infty$, *and* $\sum_{n=1}^\infty c_n < \infty$. *Then the limit* $\lim_{n\to\infty} a_n$ *exists.*

3 Main results

Theorem 3.1 *Let* C *be a nonempty closed convex subset of* H, $S : C \to C$ *be a nonexpansive mapping with a nonempty fixed point set, and* $A : C \to H$ *be a L-Lipschitz continuous and monotone mapping. Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Let* $N \ge 1$ *denote some positive integer. Assume that* $\mathcal{F} := \bigcap_{m=1}^N EP(F_m) \cap VI(C,A) \cap F(S)$ *is not empty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{n,1}\}, \ldots, \{\delta_{n,N}\}$ *be real number sequences in* $(0,1)$. *Let* $\{\lambda_n\}$, $\{r_{n,1}\}, \ldots,$ *and* $\{r_{n,N}\}$ *be positive real number sequences. Let* $\{e_n\}$ *be a bounded sequence in* H. *Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n S \operatorname{Proj}_C(\sum_{m=1}^N \delta_{n,m} z_{n,m} - \lambda_n A y_n) + \gamma_n e_n, \quad n \ge 1, \\ y_n = \operatorname{Proj}_C(\sum_{m=1}^N \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^N \delta_{n,m} z_{n,m}), \end{cases}$$

where $z_{n,m}$ *is such that*

$$F_m(z_{n,m}, z) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \ge 0, \quad \forall z \in C, \forall m \in \{1, 2 \ldots, N\}.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{n,1}\}, \ldots, \{\delta_{n,N}\}$, $\{\lambda_n\}$, $\{r_{n,1}\}, \ldots,$ *and* $\{r_{n,N}\}$ *satisfy the following restrictions*:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,
(b) $0 < a \le \beta_n \le b < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(c) $\sum_{m=1}^{N} \delta_{n,m} = 1$ *and* $0 < c \le \delta_{n,m} \le 1$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $d \le \lambda_n \le e$, *where* $d, e \in (0, 1/L)$.

Then the sequence $\{x_n\}$ *weakly converges to some point* $\bar{x} \in \mathcal{F}$.

Proof Put $u_n = \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n)$ and $v_n = \sum_{m=1}^{N} \delta_{n,m} z_{n,m}$. Letting $p \in \mathcal{F}$, we see from Lemma 2.1 that

$$\begin{aligned} \|u_n - p\|^2 &\le \|v_n - \lambda_n A y_n - p\|^2 - \|v_n - \lambda_n A y_n - u_n\|^2 \\ &= \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n \langle A y_n, p - u_n\rangle \\ &= \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n \big(\langle A y_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle \\ &\quad + \langle A y_n, y_n - u_n\rangle\big) \\ &\le \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n \langle A y_n, y_n - u_n\rangle \\ &= \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle. \end{aligned} \tag{3.1}$$

Notice that A is L-Lipschitz continuous and $y_n = \mathrm{Proj}_C(v_n - \lambda_n A v_n)$. It follows that

$$\begin{aligned} &\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle \\ &\quad = \langle v_n - \lambda_n A v_n - y_n, u_n - y_n\rangle + \langle \lambda_n A v_n - \lambda_n A y_n, u_n - y_n\rangle \\ &\quad \le \lambda_n L \|v_n - y_n\| \|u_n - y_n\|. \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain that

$$\begin{aligned} \|u_n - p\|^2 &\le \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\lambda_n L \|v_n - y_n\| \|u_n - y_n\| \\ &\le \|v_n - p\|^2 + \big(\lambda_n^2 L^2 - 1\big) \|v_n - y_n\|^2. \end{aligned} \tag{3.3}$$

On the other hand, we have from the restriction (c) that

$$\begin{aligned} \|v_n - p\|^2 &\le \left\| \sum_{m=1}^{N} \delta_{n,m} z_{n,m} - p \right\|^2 \\ &\le \sum_{m=1}^{N} \delta_{n,m} \|z_{n,m} - p\|^2 \\ &\le \sum_{m=1}^{N} \delta_{n,m} \|T_{r_{n,m}} x_n - p\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \big(\lambda_n^2 L^2 - 1\big) \|v_n - y_n\|^2. \tag{3.5}$$

This in turn implies from the restriction (d) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\beta_n\|Su_n-p\|^2+\gamma_n\|e_n-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|u_n-p\|^2+\gamma_n\|e_n-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+\beta_n\bigl(\|x_n-p\|^2+\bigl(\lambda_n^2L^2-1\bigr)\|v_n-y_n\|^2\bigr)+\gamma_n\|e_n-p\|^2 \\
&\le \|x_n-p\|^2+\beta_n\bigl(\lambda_n^2L^2-1\bigr)\|v_n-y_n\|^2+\gamma_n\|e_n-p\|^2 \\
&\le \|x_n-p\|^2+\gamma_n\|e_n-p\|^2.
\end{aligned} \tag{3.6}$$

It follows from Lemma 2.7 that the $\lim_{n\to\infty}\|x_n-p\|$ exists. This in turn shows that $\{x_n\}$ is bounded. It follows from (3.6) that

$$\beta_n\bigl(1-\lambda_n^2L^2\bigr)\|v_n-y_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|e_n-p\|^2.$$

This implies from the restrictions (b) and (d) that

$$\lim_{n\to\infty}\|v_n-y_n\|=0. \tag{3.7}$$

Notice that

$$\begin{aligned}
\|y_n-u_n\| &= \bigl\|\operatorname{Proj}_C(v_n-\lambda_nAv_n)-\operatorname{Proj}_C(v_n-\lambda_nAy_n)\bigr\| \\
&\le \bigl\|(v_n-\lambda_nAv_n)-(v_n-\lambda_nAy_n)\bigr\| \\
&\le \lambda L\|v_n-y_n\|.
\end{aligned}$$

It follows from (3.7) that

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{3.8}$$

In view of

$$\|v_n-u_n\|\le\|v_n-y_n\|+\|y_n-u_n\|,$$

we see from (3.7) and (3.8) that

$$\lim_{n\to\infty}\|v_n-u_n\|=0. \tag{3.9}$$

Notice that

$$\begin{aligned}
\|z_{n,m}-p\|^2 &= \|T_{r_{n,m}}x_n-T_{r_{n,m}}p\|^2 \\
&\le \langle T_{r_{n,m}}x_n-T_{r_{n,m}}p, x_n-p\rangle \\
&= \langle z_{n,m}-p, x_n-p\rangle \\
&= \frac{1}{2}\bigl(\|z_{n,m}-p\|^2+\|x_n-p\|^2-\|z_{n,m}-x_n\|^2\bigr), \quad \forall 1\le m\le N.
\end{aligned}$$

This implies that

$$\|z_{n,m} - p\|^2 \le \|x_n - p\|^2 - \|z_{n,m} - x_n\|^2, \quad \forall 1 \le m \le N. \tag{3.10}$$

In view of (3.10) and $v_n = \sum_{m=1}^N \delta_{n,m} z_{n,m}$, where $\sum_{m=1}^N \delta_{n,m} = 1$, we see from Lemma 2.4 that

$$\begin{aligned}
\|v_n - p\|^2 &\le \sum_{m=1}^N \delta_{n,m} \|z_{n,m} - p\|^2 \\
&\le \sum_{m=1}^N \delta_{n,m} \big(\|x_n - p\|^2 - \|z_{n,m} - x_n\|^2\big) \\
&= \|x_n - p\|^2 - \sum_{m=1}^N \delta_{n,m} \|z_{n,m} - x_n\|^2.
\end{aligned} \tag{3.11}$$

In view of (3.3), we obtain from the restriction (d) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n \|x_n - p\|^2 + \beta_n \|Su_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|v_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\
&\le \|x_n - p\|^2 - \beta_n \sum_{m=1}^N \delta_{n,m} \|z_{n,m} - x_n\|^2 + \gamma_n \|e_n - p\|^2.
\end{aligned}$$

It follows that

$$\beta_n \delta_{n,m} \|z_{n,m} - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n \|e_n - p\|^2.$$

In view of the restrictions (b) and (c), we find that

$$\lim_{n \to \infty} \|z_{n,m} - x_n\| = 0. \tag{3.12}$$

Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ. It follows from (3.12) that $\{z_{n_i,m}\}$ converges weakly to ξ for each $1 \le m \le N$. Next, we show that $\xi \in EP(F_m)$ for each $1 \le m \le N$. Since $z_{n,m} = T_{r_{n,m}} x_n$, we have

$$F_m(z_{n,m}, z) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \ge 0, \quad \forall z \in C.$$

From the assumption (A2), we see that

$$\frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \ge F_m(z, z_{n,m}), \quad \forall z \in C.$$

Replacing n by n_i, we arrive at

$$\left\langle z - z_{n_i,m}, \frac{z_{n_i,m} - x_{n_i}}{r_{n_i,m}} \right\rangle \ge F_m(z, z_{n_i,m}), \quad \forall z \in C.$$

In view of the assumption (A4), we get from (3.12) that

$$F_m(z,\xi) \leq 0, \quad \forall z \in C.$$

For t_m with $0 < t_m \leq 1$ and $z \in C$, let $z_{t_m} = t_m z + (1 - t_m)\xi$ for each $1 \leq m \leq N$. Since $z \in C$ and $\xi \in C$, we have $z_{t_m} \in C$ for each $1 \leq m \leq N$. It follows that $F_m(z_{t_m},\xi) \leq 0$ for each $1 \leq m \leq N$. Notice that

$$0 = F_m(z_{t_m}, z_{t_m}) \leq t_m F_m(z_{t_m}, z) + (1 - t_m)F_m(z_{t_m}, \xi) \leq t_m F_m(z_{t_m}, z), \quad \forall 1 \leq m \leq N,$$

which yields that

$$F_m(z_{t_m}, z) \geq 0, \quad \forall z \in C.$$

Letting $t_m \downarrow 0$ for each $1 \leq m \leq N$, we obtain from the assumption (A3) that

$$F_m(\xi, z) \geq 0, \quad \forall z \in C.$$

This implies that $\xi \in EP(F_m)$ for each $1 \leq m \leq N$. This proves that $\xi \in \bigcap_{m=1}^N EP(F_m)$.

Next, we show that $\xi \in VI(C,A)$. In fact, let T be the maximal monotone mapping defined by

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x,y) \in G(T)$, we have $y - Ax \in N_C x$. So, we have $\langle x - m, y - Ax\rangle \geq 0$, for all $m \in C$. On the other hand, we have $u_n = \mathrm{Proj}_C(v_n - \lambda_n A y_n)$. We obtain that

$$\langle v_n - \lambda_n A y_n - u_n, u_n - x\rangle \geq 0$$

and hence

$$\left\langle x - u_n, \frac{u_n - v_n}{\lambda_n} + Ay_n \right\rangle \geq 0.$$

In view of the monotonicity of A, we see that

$$\begin{aligned} \langle x - u_{n_i}, y\rangle &\geq \langle x - u_{n_i}, Ax\rangle \\ &\geq \langle x - u_{n_i}, Ax\rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\ &= \langle x - u_{n_i}, Ax - Au_{n_i}\rangle + \langle x - u_{n_i}, Au_{n_i} - Ay_{n_i}\rangle \\ &\quad - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle x - u_{n_i}, Au_{n_i} - Ay_{n_i}\rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \tag{3.13}$$

On the other hand, we see that

$$\|v_n - x_n\| \le \sum_{m=1}^{N} \delta_{n,m}\|z_{n,m} - x_n\|.$$

It follows from (3.12) that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.14}$$

Notice that

$$\|u_n - x_n\| \le \|u_n - v_n\| + \|v_n - x_n\|.$$

Combining (3.9) with (3.14), we arrive at

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.15}$$

This in turn implies that $u_{n_i} \rightharpoonup \xi$. It follows from (3.13) that $\langle x - \xi, y\rangle \ge 0$. Notice that T is maximal monotone and hence $0 \in T\xi$. This shows from Lemma 2.3 that $\xi \in VI(C,A)$.

Next, we show that $\xi \in F(S)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we put $\lim_{n\to\infty} \|x_n - p\| = d > 0$. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \big\|(1-\beta_n)\big(x_n - p + \gamma_n(e_n - x_n)\big) + \beta_n\big(Su_n - p + \gamma_n(e_n - x_n)\big)\big\| = d.$$

Notice that

$$\begin{aligned}\big\|Su_n - p + \gamma_n(e_n - x_n)\big\| &\le \|Su_n - p\| + \gamma_n\|e_n - x_n\|\\ &\le \|u_n - p\| + \gamma_n\|e_n - x_n\|\\ &\le \|x_n - p\| + \gamma_n\|e_n - x_n\|.\end{aligned}$$

This shows that

$$\limsup_{n\to\infty}\big\|Su_n - p + \gamma_n(e_n - x_n)\big\| \le d.$$

On the other hand, we have

$$\limsup_{n\to\infty}\big\|x_n - p + \gamma_n(e_n - x_n)\big\| \le d.$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} \|x_n - Su_n\| = 0. \tag{3.16}$$

In view of

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Su_n\| + \|Su_n - x_n\|\\ &\le \|x_n - u_n\| + \|Su_n - x_n\|,\end{aligned}$$

we find from (3.15) and (3.16) that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

This implies from Lemma 2.6 that $\xi \in F(S)$. This completes the proof that $\xi \in \mathcal{F}$.

Finally, we show that the whole sequence $\{x_n\}$ weakly converges to ξ. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ', where $\xi' \neq \xi$. In the same way, we can show that $\xi' \in \mathcal{F}$. Since the space H enjoys Opial's condition, we, therefore, obtain that

$$\begin{aligned} d &= \liminf_{i\to\infty} \|x_{n_i} - \xi\| < \liminf_{i\to\infty} \|x_{n_i} - \xi'\| \\ &= \liminf_{j\to\infty} \|x_j - \xi'\| < \liminf_{j\to\infty} \|x_j - \xi\| = d. \end{aligned}$$

This is a contradiction. Hence, $\xi = \xi'$. This completes the proof. □

If $N = 1$, then Theorem 3.1 is reduced to the following.

Corollary 3.2 *Let C be a nonempty closed convex subset of H, $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, and $A : C \to H$ be a L-Lipschitz continuous and monotone mapping. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Assume that $\mathcal{F} := EP(F) \cap VI(C,A) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in* $(0,1)$. *Let $\{\lambda_n\}$, $\{r_n\}$ be positive real number sequences. Let $\{e_n\}$ be a bounded sequence in H. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n S \operatorname{Proj}_C(z_n - \lambda_n A y_n) + \gamma_n e_n, \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(z_n - \lambda_n A z_n), \end{cases}$$

where z_n is such that

$$F(z_n, z) + \frac{1}{r_n}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\lambda_n\}$, $\{r_n\}$ satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,

(b) $0 < a \leq \beta_n \leq b < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(c) $\liminf_{n\to\infty} r_n > 0$ *and* $c \leq \lambda_n \leq d$, *where* $c, d \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

If $S = I$, where I stands for the identity mapping, then Theorem 3.1 is reduced to the following.

Corollary 3.3 *Let C be a nonempty closed convex subset of H and $A : C \to H$ be a L-Lipschitz continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $N \geq 1$ denote some positive integer. Assume that $\mathcal{F} := \bigcap_{m=1}^{N} EP(F_m) \cap VI(C,A)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{n,1}\}, \ldots, \{\delta_{n,N}\}$ be real number sequences in* $(0,1)$.

Let $\{\lambda_n\}$, $\{r_{n,1}\},\dots,$ and $\{r_{n,N}\}$ be positive real number sequences. Let $\{e_n\}$ be a bounded sequence in H. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n \operatorname{Proj}_C(\sum_{m=1}^N \delta_{n,m} z_{n,m} - \lambda_n A y_n) + \gamma_n e_n, \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(\sum_{m=1}^N \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^N \delta_{n,m} z_{n,m}), \end{cases}$$

where $z_{n,m}$ is such that

$$F_m(z_{n,m}, z) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \quad \forall z \in C, \forall m \in \{1, 2 \dots, N\}.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{n,1}\},\dots,\{\delta_{n,N}\}$, $\{\lambda_n\}$, $\{r_{n,1}\},\dots,$ and $\{r_{n,N}\}$ satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,
(b) $0 < a \leq \beta_n \leq b < 1$ *and* $\sum_{n=1}^\infty \gamma_n < \infty$;
(c) $\sum_{m=1}^N \delta_{n,m} = 1$ *and* $0 < c \leq \delta_{n,m} \leq 1$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $d \leq \lambda_n \leq e$, *where* $d, e \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

If $F_m(x,y) \equiv 0$ for all $x, y \in C$ and $r_{n,m} \equiv 1$, then Theorem 3.1 is reduced to the following.

Corollary 3.4 *Let C be a nonempty closed convex subset of H, $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, and $A : C \to H$ be a L-Lipschitz continuous and monotone mapping. Assume that $\mathcal{F} := VI(C, A) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{e_n\}$ be a bounded sequence in H. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n S \operatorname{Proj}_C(\operatorname{Proj}_C x_n - \lambda_n A y_n) + \gamma_n e_n, \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(\operatorname{Proj}_C x_n - \lambda_n A \operatorname{Proj}_C x_n). \end{cases}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,
(b) $0 < a \leq \beta_n \leq b < 1$ *and* $\sum_{n=1}^\infty \gamma_n < \infty$;
(c) $c \leq \lambda_n \leq d$, *where* $c, d \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

4 Applications

Theorem 4.1 *Let $S : H \to H$ be a nonexpansive mapping with a nonempty fixed point set and $A : H \to H$ be a L-Lipschitz continuous and monotone mapping. Assume that $\mathcal{F} := A^{-1}(0) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{e_n\}$ be a bounded sequence in H. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$x_1 \in H, \qquad x_{n+1} = \alpha_n x_n + \beta_n S\big(x_n - \lambda_n A(x_n - \lambda_n A x_n)\big) + \gamma_n e_n, \quad n \geq 1.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *satisfy the following restrictions*:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,
(b) $0 < a \le \beta_n \le b < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(c) $c \le \lambda_n \le d$, *where* $c, d \in (0, 1/L)$.

Then the sequence $\{x_n\}$ *weakly converges to some point* $\bar{x} \in \mathcal{F}$.

Proof Put $F_m(x,y) \equiv 0$ for all $x, y \in C$ and $r_{n,m} \equiv 1$. Notice that $A^{-1}(0) = VI(H, A)$ and $P_H = I$, we easily find from Theorem 3.1 the desired conclusion. □

Next, we consider the common zero point problem of two monotone mappings.

Theorem 4.2 *Let* $B : H \to 2^H$ *a maximal monotone mapping and* $A : H \to H$ *be a L-Lipschitz continuous and monotone mapping. Assume that* $\mathcal{F} := A^{-1}(0) \cap B^{-1}(0)$ *is not empty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be real number sequences in* $(0, 1)$. *Let* $\{\lambda_n\}$ *be a positive real number sequence. Let* $\{e_n\}$ *be a bounded sequence in* H. *Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in H, \qquad x_{n+1} = \alpha_n x_n + \beta_n J_r^B \bigl(x_n - \lambda_n A(x_n - \lambda_n A x_n)\bigr) + \gamma_n e_n, \quad n \ge 1,$$

where J_r^B *stands for the resolvent of* B *for each* $r > 0$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *satisfy the following restrictions*:

(a) $\alpha_n + \beta_n + \gamma_n = 1$,
(b) $0 < a \le \beta_n \le b < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(c) $c \le \lambda_n \le d$, *where* $c, d \in (0, 1/L)$.

Then the sequence $\{x_n\}$ *weakly converges to some point* $\bar{x} \in \mathcal{F}$.

Proof Put $F_m(x,y) \equiv 0$ for all $x, y \in C$ and $r_{n,m} \equiv 1$. Notice that $A^{-1}(0) = VI(H, A)$, $F(J_r^B) = B^{-1}(0)$, and $P_H = I$, we easily find from Theorem 3.1 the desired conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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